ON THE SYNTHESIS OF OPTIMAL DAMPERS

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Some problems of optimal damping theory are considered. It is shown that these problems can be stated in the form of the variational problem of optimal systems synthesis. The necessary steady-state condition and the necessary condition for a strong minimum of the functional for the variational problem of optimal damper synthesis are described. Sample computations of dampers for systems with one degree of freedom are carried out.

1. The equation of a dampable mass with one degree of freedom can be written as follows [1]:

$$x^{\cdot\cdot} + u(x, x^{\cdot}, t) = f(t) \tag{1.1}$$

Here x is the generalized coordinate of the system, $f(t)$ is the specified external force, and $u(x, x^{\cdot}, t)$ is a function representing the damper characteristic. This function is usually subject to the limitation

$$|u(x, x^{\cdot}, t)| \leqslant U_0 \tag{1.2}$$

Let us consider the segment $[t_0, T]$, the initial conditions

$$x(t_0) = x_0, \qquad x^{\cdot}(t_0) = x^{\cdot}_0 \tag{1.3}$$

and the functional

$$J = \max|x(t)| \qquad (t \in [t_0, T]) \tag{1.4}$$

The optimal damping problem can be formulated as follows: from among the functions $x(t)$ and $u(x, x^{\cdot}, t)$ satisfying Eq. (1.1) and inequality (1.2) on the segment $[t_0, T]$ and conditions (1.3) at the left-hand end of this segment we are to find those which minimize functional (1.4).

Since the condition(*)

$$x^{\cdot}(t_i) = 0, \qquad t_i \in [t_0, T] \tag{1.5}$$

isolates the extremal points of the function $x(t)$, instead of Expression (1.4) we can consider the functional

$$J = x^2(t_i) \tag{1.6}$$

The above problem then assumes a form which is a special case of the following general problem of synthesis of optimal systems whose functional depend on intermediate coordinate values [2].

From among the vector-functions $x(t) = \{x_1(t), \ldots, x_n(t)\}$ and from the piecewise-continuous vector functions $u(x, t) = \{u_1(x, t), \ldots, u_m(x, t)\}$ satisfying the differential equations

$$g_s = x_s^{\cdot} - f_s(x, u, t) = 0 \qquad (s = 1, \ldots, n) \tag{1.7}$$

and the finite relations

$$\Psi_k = \Psi_k(u, t) = 0 \qquad (k = 1, \ldots, r < m) \tag{1.8}$$

on the segment $[t_0, T]$ and the conditions (1.9)

$$\varphi_l = \varphi_l[x(t_0), t_0, x(t_1), t_1, \ldots, x(T), T] = 0 \qquad (l = 1, \ldots, p \leqslant (q+1)(n+1) - 1)$$

at the ends and intermediate points $t = t_i$ ($i = 1, \ldots, q-1$) of this segment, we are to find those which minimize the functional

*) The case where $x^{\cdot}(t) = 0$ on the segment $[t_1, t_2]$ requires special consideration

$$J = g\,[x(t_0),\, t_0,\, x(t_1),\, t_1, \ldots, x(T),\, T] + \int_{t_0}^{T} f_0(x,\, u,\, t)\,dt \tag{1.10}$$

This problem differs from that of constructing optimal processes in that we are required to find optimal laws $u(x, t)$ rather than $u(t)$.

2. The necessary steadystate condition for the functional I is of the form [2]

$$\Delta I = 0 \tag{2.1}$$

Here ΔI denotes the total variation of the functional I represented by the relation

$$I = \varphi + \int_{t_0}^{T} \Big[\sum_{s=1}^{n} \lambda_s x_s^{\cdot} - H\Big]\,dt \tag{2.2}$$

$$\varphi = g + \sum_{l=1}^{p} \rho_l \varphi_l, \qquad H = -f_0 + \sum_{s=1}^{n} \lambda_s f_s + \sum_{k=1}^{r} \mu_k \psi_k \tag{2.3}$$

where $\lambda_s(x, t)$, $\mu_k(x, t)$, ρ_l are indefinite Lagrange multipliers. In computing the variation ΔI one must take account of the dependence of the functions $u_k(x, t)$, $\lambda_s(x, t)$, $\mu_k(x, t)$ on the coordinates x_s. Then, for example, the total variation Δu_k of the control $u_k(x, t)$ can be written as

$$\Delta u_k = \delta u_k + \sum_{s=1}^{n} \frac{\partial u_k}{\partial x_s}\,\delta x_s$$

where δu_k represents the partial variation of the function u_k.

Carrying out the manipulations and discussion typical of the calculus of variations [3], we obtain the steady-state condition in expanded form. It consists

of the partial differential equations

$$\frac{\partial \lambda_s}{\partial t} + \sum_{\alpha=1}^{n} \frac{\partial \lambda_s}{\partial x_\alpha} f_\alpha + \frac{\partial H}{\partial x_s} = 0 \qquad (s = 1, \ldots, n) \tag{2.4}$$

of the relations

$$\frac{\partial H}{\partial u_k} = 0 \qquad (k = 1, \ldots, m) \tag{2.5}$$

of the end conditions

$$\begin{aligned} &\frac{\partial \varphi}{\partial t_0} + (H)_{t_0} = 0, \qquad (\lambda_s)_{t_0} - \frac{\partial \varphi}{\partial x_s(t_0)} = 0 \\ &\frac{\partial \varphi}{\partial T} - (H)_T = 0, \qquad (\lambda_s)_T + \frac{\partial \varphi}{\partial x_s(T)} = 0 \end{aligned} \qquad (s = 1, \ldots, n) \tag{2.6}$$

of the Erdmann-Weierstrass conditions for the discontinuity points $t = t^*$ of the parameters of the control $u_k(t)$

$$(H)_{t^*-0} - (H)_{t^*+0} = 0, \qquad (\lambda_s)_{t^*-0} - (\lambda_s)_{t^*+0} = 0 \qquad (s = 1, \ldots, n) \tag{2.7}$$

and of the Erdmann-Weierstrass conditions at the points $t = t_i$,

$$\frac{\partial \varphi}{\partial t_i} - (H)_{t_i-0} + (H)_{t_i+0} = 0, \; (\lambda_s)_{t_i-0} - (\lambda_s)_{t_i+0} + \frac{\partial \varphi}{\partial x_s(t_i)} = 0$$

$$(i = 1, \ldots, q-1;\; s = 1, \ldots, n) \tag{2.8}$$

In constructing the optimal functions $u_k(x, t)$ it is also necessary to make use of Eqs. (1.7) and (1.8), the conditions of continuity of the coordinates

$$x_s(t^* - 0) - x_s(t^* + 0) = 0, \qquad x_s(t_i - 0) - x_s(t_i + 0) = 0$$

$$(s = 1, \ldots, n;\; i = 1, \ldots, q-1) \tag{2.9}$$

and Eqs. (1.9). In order for the necessary Weierstrass condition of a strong minimum of the functional J to be fulfilled, one must guarantee fulfillment of the steady-state condition and the Weierstrass inequality; the latter takes the form

$$H\,(x,\, u,\, \lambda,\, \mu,\, t) \geqslant H\,(x,\, U,\, \lambda,\, \mu,\, t) \tag{2.10}$$

Here $u(x, t)$ are functions which minimize the functional J, while $U(x, t) \neq u(x, t)$ are any admissible functions.

Comparison of Eqs. (2.4) with the corresponding equations of the system $\lambda_s^{\cdot} + \partial H/\partial x_s$ used in the construction of optimal processes [2] shows that in solving the synthesis problem we are obliged to deal with partial differential equations whose characteristics are curves satisfying the equations of the variational problem of optimal process construction.

In the case where the functions f_s and φ_k do not depend on time explicitly, the controls u_k and multipliers λ_s and μ_k must also be found in the form of time-independent functions. Eqs. (2.4) then become

$$\sum_{\alpha=1}^{n} \frac{\partial \lambda_s}{\partial x_\alpha} f_\alpha + \frac{\partial H}{\partial x_s} = 0 \qquad (s = 1, \ldots, n) \tag{2.11}$$

Eqs. (2.4) or (2.11) can also be used in solving problems of synthesis of optimal systems described by equations with discontinuous right-hand sides or of systems with bounded coordinates. The only difference occurs in relations (2.8). The corresponding conditions are given in [4 and 5].

3. Let us consider the problem of optimal damping of a mass with one degree of freedom We assume that an instantaneous pulse is applied to the mass at the initial instant $t = t_0 = 0$. Then in Eq. (1.1) we have $f = 0$, and the initial conditions can be written as $x(0) = 0$, $x^{\cdot}(0) = x_0^{\cdot}$. On introducing the notation $x_1 = x$, $x_2 = x^{\cdot}$, instead of (1.1) we have

$$x_1^{\cdot} = x_2, \qquad x_2^{\cdot} = -u \tag{3.1}$$

while the initial conditions take the form

$$x_1(0) = 0, \qquad x_2(0) = x_{20} \qquad (x_{20} = x_0^{\cdot}) \tag{3.2}$$

We convert to an open range of admissible variations of the controls by constructing the relation

$$\psi = u^2 + v^2 - U_0^2 = 0 \tag{3.3}$$

which contains the additional parameter v.

The optimal damping problem will now be considered in two formulations. In the first of these the functional is written in the form

$$J = x_1^2(T) \tag{3.4}$$

and fulfillment of the Eq.

$$x_2(T) = 0 \tag{3.5}$$

is required.

Here we seek the minimum of the extremum of the function $x_1(t)$ for $t \in [t_0, T]$ under the condition that no limitations are imposed on the motion of the system after the extremum has been attained.

The second formulation is based on the functional

$$J = x_1^2(t_1) \tag{3.6}$$

and fulfillment of the conditions

$$x_2(t_1) = x_1(T) = x_2(T) = 0 \tag{3.7}$$

is required.

Upon termination of the process the system must arrive at the origin of the coordinates.

In both cases the function H is of the form

$$H = \lambda_1 x_2 - \lambda_2 u + \mu\,[u^2 + v^2 - U_0^2]$$

Hence, we obtain the following Eqs. on the basis of (2.11) and (2.5):

$$\frac{\partial \lambda_1}{\partial x_1} x_2 - \frac{\partial \lambda_1}{\partial x_2} u = 0, \qquad \frac{\partial \lambda_2}{\partial x_1} x_2 - \frac{\partial \lambda_2}{\partial x_2} u + \lambda_1 = 0 \tag{3.8}$$

$$-\lambda_2 + 2\mu u = 0, \qquad 2\mu v = 0 \tag{3.9}$$

Inequality (2.10) can be written in the form $\lambda_2 u \leqslant \lambda_2 U$. On the basis of this relation we have

$$u(x_1, x_2) = -U_0 \operatorname{sign} \lambda_2(x_1, x_2) \tag{3.10}$$

This relation must be fulfilled in the optimal system. Hence, the problem of synthesis of an optimal damper reduces to constructing the multiplier $\lambda_2 = \lambda_2(x_1, x_2)$. Turning now to the determination of the latter, we construct the function φ. In the first of the above problems it is of the form

$$\varphi = x_1^2(T) + \rho_1 x_1(t_0) + \rho_2[x_2(t_0) - x_{20}] + \rho_3 x_2(T) + \rho_4 t_0$$

while the end conditions can be written as

$$(\lambda_1)_{t_0} = \rho_1, \quad (\lambda_2)_{t_0} = \rho_2, \quad (\lambda_1)_T = -2x_1(T), \quad (\lambda_2)_T = -\rho_3$$

$$(H)_{t_0} = -\rho_4, \quad (H)_T = 0 \tag{3.11}$$

Conditions (2.7) indicate that λ_i and H are continuous on the segment $[t_0, T]$. Making use of the first and second conditions of (3.11) we find that instead of ρ_1 and ρ_2 we can attempt to find the values of $(\lambda_1)_{t_0}$ and $(\lambda_2)_{t_0}$. The latter of conditions (3.11) yields $(H)_T = 0$. Allowance for the fact that the problem contains the first integral $H = \text{const}$ yields the important equation

$$H \equiv 0, \qquad t \in [t_0, T]$$

Making use of condition (3.5), we obtain the relation

$$(\lambda_2)_T = 0 \tag{3.12}$$

The general solution of Eqs. (3.8) for $u = \text{const}$ is of the form [6] (3.13)

$$\lambda_1(x_1, x_2) = \Phi_1\left(x_1 + \frac{x_2^2}{2u}\right), \qquad \lambda_2(x_1, x_2) = \Phi_2\left(x_1 + \frac{x_2^2}{2u}\right) + \frac{x_2}{u}\Phi_1\left(x_1 + \frac{x_2^2}{2u}\right)$$

Here Φ_1 and Φ_2 are arbitrary functions. In determining them we make use the third condition of (3.11) and Eq. (3.12). We then have

$$\Phi_1 = -2x_1(T), \qquad \Phi_2 = 0$$

Hence, for the function λ_2 we have

$$\lambda_2(x_1, x_2) = -2x_1(T)x_2 / u$$

This formula solves the problem of optimal damper synthesis.

On simplifying it we note that on the basis of Eqs. (3.1) we may establish the validity of the Eq. $\operatorname{sign} x_1(T) = \operatorname{sign} u$ provided that u is continuous and has one of the values $u = \pm U_0$. But then $\operatorname{sign} \lambda_2 = -\operatorname{sign} x_2$, and the synthesizing law is the simple formula

$$u(x_1, x_2) = U_0 \operatorname{sign} x_2 \tag{3.14}$$

If we assume that the damper ceases to function at the instant $t = T$, so that $u = 0$ for $t > T$, then for $t > T$ we have $x_1 = x_1(T)$, $x_2 = 0$. We may, of course, to supply the system with an additional damping device which returns it to the origin $x_1 = x_2 = 0$.

In the second of the above problems we assume that both ends of the comparison curves are fixed. The function φ is then given by

$$\varphi = x_1^2(t_1) + \rho_1 x_2(t_1) + \rho_2 x_1(t_0) + \rho_3 x_2(t_0) + \rho_4 t_0 + {}_5 x_1(T) + \rho_6 x_2(T)$$

Hence, the end conditions can be written as

$$(\lambda_1)_{t_0} = \rho_2, \quad (\lambda_2)_{t_0} = \rho_3, \quad (\lambda_1)_T = -\rho_5, \quad (\lambda_2)_T = -\rho_6, \quad (H)_{t_0} = -\rho_4 \tag{3.15}$$

Eqs. (2.7) are fulfilled at the discontinuity point of the parameter u, while the Erdmann-Weierstrass conditions for $t = t_1$ are given by Eqs.

$$(\lambda_1)_{t_1-0} - (\lambda_1)_{t_1+0} + x_1(t_1) = 0, \quad (\lambda_2)_{t_1-0} - (\lambda_2)_{t_1+0} + \rho_1 = 0$$

$$(H)_{t_1-0} - (H)_{t_1+0} = 0 \tag{3.16}$$

Analysis of the solutions of Eqs. (3.1) shows that if the piecewise-continuous function $u(t)$ has a discontinuity at $t = t_1$, then $x_1(t)$ does not have a maximum at this point. However, by virtue of (3.7) and (3.16) at $t = t_1$ we have $(\lambda_2 u)_{t_1-0} = (\lambda_2 u)_{t_1+0}$. Hence, the multiplier λ_2 is continuous at $t = t_1$.

In constructing the synthesizing function we make use of the fact that during optimal operation the representing point arrives at the origin along the curve $2U_0 x_1 = x_2^2$ for $u = -U_0$ or along the curve $-2U_0 x_1 = x_2^2$ for $u = +U_0$. The equation $\lambda_2 = 0$ must be fulfilled on these curves.

Now, making use of Formulas (3.13) in the case where $u = U_0$ for $t \in [t_0, t^*]$, we obtain Expressions

$$\lambda_2 = \Phi_1(\xi_1)\left[\frac{x_2}{U_0} + \frac{1}{\sqrt{U_0}}\sqrt{\xi_1}\right] - \frac{x_2}{U_0} \qquad (t \in [t_0, t_1])$$

$$\lambda_2 = \Phi_1(\xi_1)\left[\frac{x_2}{U_0} + \frac{1}{\sqrt{U_0}}\sqrt{\xi_1}\right] \qquad (t \in [t_1, t^*]) \qquad \left(\xi_1 = x_1 + \frac{x_2^2}{2U_0}\right) \quad (3.17)$$

This requires fulfillment of the inequality $\Phi_1 < 0$. In the case where the parameter u is negative for $t \in [t_0, t^*]$ and $u = -U_0$, these equations are replaced by

$$\lambda_2 = \Phi_1(\xi_2)\left[-\frac{x_2}{U_0} + \frac{1}{\sqrt{U_0}}\sqrt{\xi_2}\right] + \frac{x_2}{U_0} \qquad (t \in [t_0, t_1])$$

$$\lambda_2 = \Phi_1(\xi_2)\left[-\frac{x_2}{U_0} + \frac{1}{\sqrt{U_0}}\sqrt{\xi_2}\right] \qquad (t \in [t_1, t^*])$$

$$(\xi_2 = x_1 - x_2^2/2U_0) \quad (\Phi_1 > 0) \qquad (3.18)$$

In simplifying the functions just constructed we recall that $\lambda_2 < 0$ if it is defined by Formulas (3.17) and $\lambda_2 > 0$ if it is determined from Formulas (3.18). It is clear that these inequalities are fulfilled if $x_2 > 0$ in the first of relations (3.17) and if $x_2 < 0$ in the first Expression of (3.18). These considerations lead to the simple equation

$$u = U_0 \operatorname{sign} x_2 \qquad (t \in [t_0, t_1)) \qquad (3.19)$$

In a similar fashion, for $t \in (t_1, t^*)$ we obtain

$$u = U_0 \operatorname{sign}\,[x_1 + x_2^2 / 2U_0 \operatorname{sign} x_2] \qquad (3.20)$$

The difficulty which we encounter in using these synthesizing functions consists in the necessity of allowing for Eq. $x_2(t_1) = 0$ which defines transition from one function to the other. It is easy to show, however, that if the parameter u for $t \in [t_0, t_1)$ is determined from (3.20), then its values coincide with the quantities given by Formula (3.19). Hence, we can take function (3.20) as the synthesizing function.

It is interesting that Eq. (3.20) defines the optimal law in the speed-of-response problem. Comparison of Formulas (3.19) and (3.14) shows that the optimal controls in the two problems coincide until fulfillment of the relation $x_1 = 0$; further changes in the control in the second problem occur in accordance with the optimal speed-of-response law.

4. Let us consider a system with one degree of freedom acted on by a long pulse. Let Eq.

$$f(t) = F \quad (t \in [t_0, \tau)), \qquad f(t) = 0 \quad (t > \tau) \qquad (4.1)$$

be fulfilled in Eq. (1.1) and let U_0 satisfy the inequality $U_0 < F$. After introducing the notation $x = x_1$, $x^\cdot = x_2$, we can rewrite the equation of problem (1.1) as

$$x_1^\cdot = x_2, \qquad x_2^\cdot = F - u(x_1, x_2), \quad x_1^\cdot = x_2, \quad x_2^\cdot = -u(x_1, x_2) \quad (4.2)$$

Their right-hand sides have a discontinuity at the point $t' = \tau$. We assume the left end of the comparison curves fixed (see relations (3.2)) and consider the problem of minimizing functional (3.4) when Eq. (3.5) is fulfilled. Constructing the function H, we have

$$H = \lambda_1 x_2 + \lambda_2 [F - u] + \mu [u^2 + v^2 - U_0^2] \quad (t \in [t_0 \tau))$$

$$H = \lambda_1 x_2 - \lambda_2 u + \mu [u^2 + v^2 - U_0^2] \qquad (t > \tau)$$

Equations of the problem for $t \in [t_0, \tau)$ can be written as

$$\frac{\partial \lambda_1}{\partial x_1} x_2 + \frac{\partial \lambda_1}{\partial x_2}[F - u] = 0, \qquad \frac{\partial \lambda_2}{\partial x_1} x_2 + \frac{\partial \lambda_2}{\partial x_2}(F - u) + \lambda_1 = 0 \qquad (4.3)$$

$$-\lambda_1 + 2\mu u = 0, \qquad 2\mu v = 0 \qquad (4.4)$$

For $t > \tau$ the corresponding equations are of the form (3.8), (3.9). Analysis of the Weierstrass inequality and relations (3.9) and (4.4) indicates that relation (3.10) must be fulfilled during optimal operation. The boundary conditions for the functions λ_1 and λ_2 can be found using Formulas (2.16) of [4]. If we substitute the function φ given by

$$\varphi = x_1^2(T) + \rho_1 x_1(t_0) + \rho_2 [x_2(t_0) - x_{20}] + \rho_3 x_2(T) + \rho_4 t_0$$

and the function $\vartheta = t' - \tau$ into this problem, we find that the boundary conditions can be written in the form (3.11). The Erdmann-Weierstrass conditions at the point $t = \tau$ become

$$(\lambda_1)_{\tau-0} - (\lambda_1)_{\tau+0} = 0, \qquad (\lambda_2)_{\tau-0} - (\lambda_2)_{\tau+0} = 0, \qquad (H)_{\tau-0} - (H)_{\tau+0} + \nu = 0 \quad (4.5)$$

where ν is a Lagrange multiplier.

For $t > \tau$ the multipliers λ_1 and λ_2 are determined as in the previous section. The multiplier λ_2 is given by Expression $\lambda_2 = -2x_1(T)x_2/u$.

Now, constructing the general solution of Eqs. (4.3), we obtain expressions which follow from Formulas (3.13) in the replacement of u by $U_0 - F$. The condition $(\lambda_2)_{\tau_1-0} = (\lambda_2)_{\tau_1+0}$ then makes it possible to construct the solution of the second Eq. of (4.3) in the form

$$\lambda_2(x_1, x_2) = 2\frac{F - U_0}{U_0}x_1(T)x_2$$

Hence, the synthesizing function on the entire segment $[t_0, T]$ is of the form

$$u(x_1, x_2) = U_0 \operatorname{sign} x_2$$

This result was obtained for $F > 0$. It is easy to show that it is also valid in the case $F < 0$.

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